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THERMO-PLASTIC INSTABILITY IN SIMPLE SHEAR

Y. L. BAI

Institute of Mechanics, Academia Sinica, Peking, China

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ABSTRACT

A THEORETICAL description of thermo-plastic instability in simple shear is presented in a system of equations describing plastic deformation, the first law of thermodynamics and Fourier's heat transfer rule. Both mechanical and thermodynamical parameters influence instability and it is shown that two different modes of instability may exist. One of them is dominated by thermal softening and has a characteristic time and length, connected to each other by thermal diffusion.

A criterion combining thermal softening, current stress, density, specific heat, work-hardening, thermal conductivity and current strain rate is obtained and practical implications are discussed.

NOTATION

x_i	Euler coordinates
X_i	Lagrangian coordinates
T_{ij}	Cauchy stress tensor
W_p	plastic work
q	heat
K	Tayor–Quinney coefficient
u	displacement
γ	shear strain
τ	shear stress
θ	temperature
ρ	density
h_i	heat flux
E	internal energy per unit mass
c_v	specific heat
λ	thermal conductivity
Q_0	work hardening
R_0	strain-rate hardening
P_0	thermal softening
k	wave number
α	reciprocal of characteristic time

1. INTRODUCTION

IT HAS been established that a localization of plastic flow in shear can occur, which is closely connected with the heat generated by plastic deformation. Some investigators, such as ROGERS (1979), call this "adiabatic shear instability", although the phenomenon may include heat transfer during the course of deformation. This catastrophic shear is quite significant, especially for ductile fracture.

CULVER (1973) and SPRETNAK (1968) have proposed criteria for instability based on the condition

$$\mathrm{d}\sigma = 0 \tag{1.1}$$

where σ is the current flow stress. Another unpublished analytic work, by R. J. Clifton, has been concisely cited in a paper by COSTIN, CRISMAN, HAWLEY and DUFFY (1979).

Here, from the viewpoint of instability, a theoretical analysis is derived from a system of equations describing plastic deformation, the first law of thermodynamics and Fourier's law of heat conduction, in order to obtain a comprehensive and precise picture of the instability phenomenon.

2. ASSUMPTIONS

In this section a series of assumptions is presented to clarify the problem and give the analysis a rigorous base.

Assumption 1

The relationship between the plastic work W_p and the heat q produced by it, is as follows (TAYLOR and QUINNEY, 1934)

$$q = KW_{\mathrm{p}}, \tag{2.1}$$

where $K \simeq 0.9$. The remaining portion $(1-K)W_{\mathrm{p}}$ remains latent in the metal.

Assumption 2

Because the elastic deformation energy is much smaller than that due to plastic deformation, the former may be neglected. Thus,

$$W_{\mathrm{p}} = W = \int T_{ij}\dot{x}_{i,j}\,\mathrm{d}t, \tag{2.2}$$

where T_{ij} are the components of the Cauchy stress tensor, x_i are Eulerian coordinates and the dot signifies the material (Lagrangian) time derivative.

Assumption 3

We confine ourselves to the simple geometrical configuration and deformation

$$\left.\begin{aligned} x_1 &= u(X_1, X_2) + X_1, \\ x_2 &= X_2, \\ x_3 &= X_3, \end{aligned}\right\} \tag{2.3}$$

where u is the displacement in the x_1 direction and X_i are the Lagrangian coordinates. This implies that deformation can only occur in one direction but may have a gradient in the other direction.

For the deformation, a further assumption is made. This is that the normal strains are very small, so that

$$\varepsilon = \frac{\partial u}{\partial X_1} \ll 1. \tag{2.4}$$

Shear strain is denoted as

$$\gamma = \frac{\partial u}{\partial X_2}. \tag{2.5}$$

Assumption 4

Some general assumptions concerning the constitutive relation are proposed: the material exhibits no strain-rate history effects and is incompressible and isotropic.

In the case of simple shear mentioned above, the constitutive relation can be expressed as

$$\left.\begin{aligned} T_{12} &= f(\gamma, \dot{\gamma}, \theta; \quad \textstyle\int T_{12}\, d\gamma), \\ T_{11} &= g(\gamma, \dot{\gamma}, \theta; \quad \textstyle\int T_{12}\, d\gamma), \\ T_{22} &= h(\gamma, \dot{\gamma}, \theta; \quad \textstyle\int T_{12}\, d\gamma, \end{aligned}\right\} \tag{2.6}$$

and the equation of motion implies

$$\rho \frac{\partial^2 \gamma}{\partial t^2} = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) T_{12} + \frac{\partial^2}{\partial x_1\, \partial x_2}(T_{11} + T_{22}). \tag{2.7}$$

Assumption 5

Heat conduction is governed by Fourier's law

$$h_i = -\lambda \theta_{,i}, \tag{2.8}$$

where h_i are the components of heat flux, θ is the temperature and λ is the thermal conductivity. Therefore, the energy equation becomes

$$K \dot{W}_p = \rho c_v \dot{\theta} - \lambda \Delta \theta, \tag{2.9}$$

where Δ is the Laplace operator. In the case concerned,

$$K T_{12} \frac{\partial \gamma}{\partial t} = \rho c_v \left(\frac{\partial \theta}{\partial t} + \dot{u} \frac{\partial \theta}{\partial x_1}\right) - \lambda \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) \theta. \tag{2.10}$$

Assumption 6

Similarly to the Prandtl boundary layer hypothesis, it is supposed that characteristic scales L_{x_1}, L_{x_2} satisfy

$$L_{x_1} \gg L_{x_2}. \tag{2.11}$$

This means that the variation of state variables in the x_2 direction is much steeper than that in the x_1 direction and gives, to lowest order,

$$\left.\begin{aligned}\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\right)T_{12}+\frac{\partial^2}{\partial x_1\,\partial x_2}(T_{11}+T_{22})\to\frac{\partial^2 T_{12}}{\partial x_2^2},\\ \left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\right)\theta\to\frac{\partial^2\theta}{\partial x_2^2},\\ \frac{\partial\theta}{\partial t}+\dot{u}\frac{\partial\theta}{\partial x_1}\to\frac{\partial\theta}{\partial t}.\end{aligned}\right\}\tag{2.12}$$

Finally, we obtain

$$\left.\begin{aligned}\rho\frac{\partial^2\gamma}{\partial t^2}&=\frac{\partial^2\tau}{\partial y^2},\\ K\tau\frac{\partial\gamma}{\partial t}&=\rho c_v\frac{\partial\theta}{\partial t}-\lambda\frac{\partial^2\theta}{\partial y^2},\end{aligned}\right\}\tag{2.13}$$

using the notation $\tau = T_{12}$ and $y = x_2$.

To conclude this section, there are three points that should be emphasized. Firstly, examining all of the assumptions, it can be seen that the system of equations (2.13) can deal with large shear deformation, because no limitation on shear deformation has been introduced. Secondly, the first equation in (2.13) is basically a wave equation, but the right-hand side of the second is a typical diffusion equation. These two different types of phenomena are coupled through the term $K\tau\frac{\partial\gamma}{\partial t}$. This is the distinctive feature of the phenomenon under consideration. Finally this system of equations is obviously non-linear.

HOPKINS (1972) has pointed out that the question of the nature of the mathematical structure of the equations governing plastic flow is of crucial importance. The normal treatment of the problem tends to be complicated when either a large number of low-order equations or a small number of high-order equations is involved. Nevertheless, he has outlined the mathematical structure of a set of first-order partial differential equations by the method of characteristics. As for the governing equations (2.13), because of the mathematical difficulty mentioned abve, attention here is focused on the occurrence of instability. The perturbation method, which is widely used to deal with the ccurrence of instability in fluid dynamics, for instance see LIN (1955) and LANDAU and LIFSHITZ (1959), is adapted in following sections.

Then the problem is to find the condition under which a smooth deformation process changes into catastrophe. Therefore, instead of a steady state, a smoothly developing deformation state γ_0, τ_0, θ_0 is taken as base-line which is a solution of equations (2.13). We then examine what will happen, if a perturbation is superimposed on it. It will be shown in the next section that in this simple shear case only the current state parameters and their derivatives, which are governed by the "local structure" of the constitutive relation, control the instability phenomenon, even though the constitutive relation (2.6) could have a rather general form and be rate and strain-history dependent.

3. Solution

The perturbation method is used to treat the preceding system of equations. We suppose that

$$\left.\begin{aligned} \gamma &= \gamma_0+\gamma'; \quad \gamma' \ll \gamma_0,\\ \tau &= \tau_0+\tau'; \quad \tau' \ll \tau_0,\\ \theta &= \theta_0+\theta'; \quad \theta' \ll \theta_0,\end{aligned}\right\} \tag{3.1}$$

where γ_0, τ_0, θ_0 is a solution of the system (2.13), and

$$\left.\begin{aligned} \gamma' &= \gamma_* \, e^{\alpha t+iky},\\ \tau' &= \tau_* \, e^{\alpha t+iky},\\ \theta' &= \theta_* \, e^{\alpha t+iky}.\end{aligned}\right\} \tag{3.2}$$

Then, substituting γ', τ' and θ' into equation (2.13), we obtain

$$\left.\begin{aligned} \rho\frac{\partial^2\gamma'}{\partial t^2} &= \frac{\partial^2\tau'}{\partial y^2},\\ K\tau_0\frac{\partial\gamma'}{\partial t}+K\tau'\frac{\partial\gamma_0}{\partial t} &= \rho c_v\frac{\partial\theta'}{\partial t}-\lambda\frac{\partial^2\theta'}{\partial y^2},\end{aligned}\right\} \tag{3.3}$$

or

$$\left.\begin{aligned} \rho\alpha^2\gamma_*+k^2\tau_* &= 0,\\ K\tau_0\alpha\gamma_*+K\dot{\gamma}_0\tau_*-(\rho c_v\alpha+\lambda k^2)\theta_* &= 0,\end{aligned}\right\} \tag{3.4}$$

because of the relations

$$\frac{\partial\gamma_0}{\partial t} = \frac{\mathrm{d}\gamma_0}{\mathrm{d}t}-\dot{u}\frac{\partial\gamma_0}{\partial x} \simeq \frac{\mathrm{d}\gamma_0}{\mathrm{d}t} = \dot{\gamma}_0. \tag{3.5}$$

Differentiating the constitutive relation (2.6), we obtain

$$\mathrm{d}\tau = Q_0\,\mathrm{d}\gamma+R_0\,\mathrm{d}\dot{\gamma}-P_0\,\mathrm{d}\theta, \tag{3.6}$$

where

$$\left.\begin{aligned} Q_0 &= \left(\frac{\partial\tau}{\partial\gamma}\right)_0 = \left(\frac{\partial f}{\partial\gamma}+\frac{\partial f}{\partial W_p}\tau\right)_0,\\ R_0 &= \left(\frac{\partial\tau}{\partial\dot{\gamma}}\right)_0 = \left(\frac{\partial\tau}{\partial\ln\dot{\gamma}}\right)_0\frac{1}{\dot{\gamma}_0} = \frac{R_0^*}{\dot{\gamma}_0},\\ P_0 &= -\left(\frac{\partial\tau}{\partial\theta}\right)_0,\end{aligned}\right\} \tag{3.7}$$

or

$$\tau_* = Q_0\gamma_*+R_0\alpha\gamma_*-P_0\theta_*. \tag{3.8}$$

Q_0, R_0 and P_0 are work hardening, strain-rate hardening and thermal softening respectively.

Then the homogeneous system of equations

$$\left.\begin{aligned}[\rho\alpha^2+(Q_0+R_0\alpha)k^2]\gamma_*-P_0k^2\theta_* &= 0,\\ [K\tau_0\alpha+K\dot{\gamma}_0(Q_0+R_0\alpha)]\gamma_*-[K\dot{\gamma}_0P_0+\rho c_v\alpha+\lambda k^2]\theta_* &= 0\end{aligned}\right\} \tag{3.9}$$

is obtained. Since the determinant of the coefficients should be equal to zero, this leads to

$$\rho^2c_v\alpha^3+\rho[KP_0\dot{\gamma}_0+(\lambda+c_vR_0)k^2]\alpha^2+(\lambda R_0k^2+\rho c_vQ_0-K\tau_0P_0)k^2\alpha+\lambda Q_0k^4=0. \tag{3.10}$$

This is the spectral equation. If α has a positive real root, it implies that instability is possible.

In equation (3.10), c_v, λ and R_0 are positive, generally. If there is work hardening, so that $Q_0>0$, a necessary condition for a positive root α of (3.10) is a negative coefficient of α. Hence, if $P_0=0$, no instability can occur.

Rearranging, using the dimensionless variables

$$\left.\begin{aligned}\tilde{\alpha}=\frac{\lambda\alpha}{c_vQ_0},\qquad \tilde{k}^2=\frac{\lambda^2k^2}{\rho c_v^2Q_0},\\ \mathrm{A}=\frac{c_vR_0}{\lambda},\qquad \mathrm{B}=\frac{K\tau_0P_0}{\rho c_vQ_0},\qquad \mathrm{C}=\frac{K\lambda P_0\dot{\gamma}_0}{\rho c_v^2Q_0},\end{aligned}\right\} \tag{3.11}$$

reduces the spectral equation to the form

$$\tilde{\alpha}^3+[C+(1+A)\tilde{k}^2]\tilde{\alpha}^2+[A\tilde{k}^2+1-B]\tilde{k}^2\tilde{\alpha}+\tilde{k}^4=0. \tag{3.12}$$

Now we discuss the two extreme situations:

(i) For long wavelengths ($k\to 0$), the solutions of the spectral equation (3.10) are

$$\left.\begin{aligned}\alpha&=0,\\ \alpha&=-\frac{KP_0\dot{\gamma}_0}{\rho c_v}.\end{aligned}\right\} \tag{3.13}$$

Then it is deduced that shear deformation is always stable.

(ii) For short wavelengths ($k\to\infty$), the only finite solution is

$$\alpha=-\frac{Q_0}{R_0}. \tag{3.14}$$

Shear deformation is again always stable. But we have seen that there is certainly a negative term $-K\tau_0P_0k^2\alpha$ which may lead to instability. Therefore, if instability occurs, it must occur at a special set of wavelengths or wave numbers. It is of interest, therefore, to seek the wave number k_m for which the corresponding $\alpha_m>0$ is a maximum. In addition to the spectral equation (3.12), $\tilde{k}_m$ and $\tilde{\alpha}_m$ have to satisfy the equation

$$\frac{d^2\tilde{\alpha}_m}{d\tilde{k}_m^2}=0, \tag{3.15}$$

that is,

$$\tilde{k}_m^2 = \tilde{\alpha}_m \frac{(B-1)-(A+1)\tilde{\alpha}_m}{2(A\tilde{\alpha}_m+1)}. \tag{3.16}$$

Keeping $\tilde{k}_m^2 > 0$ in mind, we arrive at an important inequality to determine the limit of the $\tilde{\alpha}_m$ value:

$$0 \leqslant \tilde{\alpha}_m \leqslant \frac{B-1}{A+1} = \tilde{\alpha}_m^*. \tag{3.17}$$

Combining both the spectral equation (3.12) and the extreme condition (3.16), we get the equation

$$F_1 = F_2 \tag{3.18}$$

to determine $\tilde{\alpha}_m$, where

$$F_1 = 4(A\tilde{\alpha}_m+1)(\tilde{\alpha}_m+C), \tag{3.19}$$

$$F_2 = [(1+A)\tilde{\alpha}_m-(B-1)]^2 = (1+A)^2(\tilde{\alpha}_m-\tilde{\alpha}_m^*)^2. \tag{3.20}$$

The last equality in (3.20) defines the parameter α_m^*.

4. Two Modes of Instability

It may be seen from the diagram of $F_{1,2}$ vs $\tilde{\alpha}_m$ (Fig. 1) that for the region $\tilde{\alpha}_m > 0$, the left branch of function F_2 and the right branch of F_1 must have an intersection between 0 and $\tilde{\alpha}_m^*$, as long as

$$B > 1+\sqrt{(4C)}. \tag{4.1}$$

This inequality is a criterion for the existence of a solution $\tilde{\alpha}_m$, and therefore is the criterion we desire. In many cases, it is true that $C \ll 1$; the criterion for instability then simplifies to

$$B > 1 \tag{4.2}$$

or

$$\frac{K\tau_0 P_0}{\rho c_v Q_0} > 1. \tag{4.3}$$

This means that the thermal softening caused by plastic work overcomes the work hardening of the material. It is very interesting to point out that whether instability occurs or not is not related to the thermal conductivity λ, strain-rate hardening R_0 and current strain rate $\dot{\gamma}_0$. However, these factors influence instability markedly in some other aspects which will be explained later.

The intersection α_m in Fig. 1 and the corresponding value of $\tilde{k}_m$ represent the most probable unstable solution. The solution $\tilde{\alpha}_m$ has the same order as $\tilde{\alpha}_m^*$.

Hence, for qualitative discussion, the value of $\tilde{\alpha}_m^*$ can be used to represent the point of intersection $\tilde{\alpha}_m$.

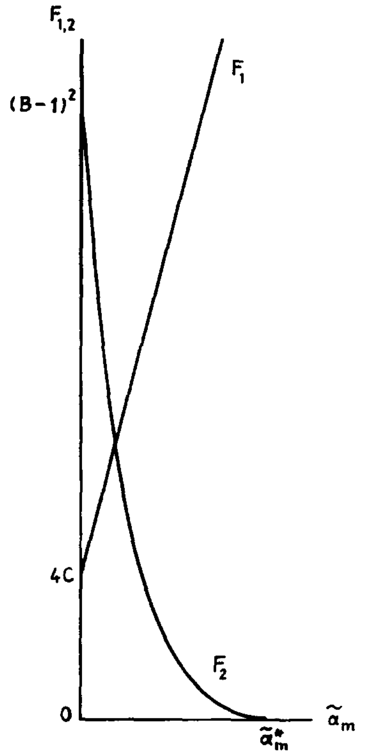

FIG. 1. Plots of the functions F_1, F_2, defined in equations (3.19) and (3.20).

If A $\gg$ 1, then

$$\tilde{\alpha}_m^* \sim \frac{\mathrm{B}-1}{\mathrm{A}} \tag{4.4}$$

and the characteristic time

$$\begin{aligned} t_c &\sim \frac{1}{\alpha_m} \sim \frac{\lambda}{c_v Q_0} \frac{1}{\tilde{\alpha}_m} \\ &\geqslant \frac{\lambda}{c_v Q_0} \frac{\mathrm{A}}{\mathrm{B}-1} \\ &\sim \frac{R_0^*}{\dot{\gamma}_0} \frac{\rho c_v}{K\tau_0 P_0 - \rho c_v Q_0}. \end{aligned} \tag{4.5}$$

It is clear that the characteristic time t_c is affected by strain-rate hardening; strain-rate and the extent to which the phenomenon concerned is past instability. Emphasis should be placed on the current strain rate $\dot{\gamma}_0$ which influences the characteristic time rather strongly. In high strain rate testing the time decreases very rapidly. This might be one of the reasons for thermo-plastic shear instability at high strain rates.

The characteristic length l_c is related to t_c by

$$\frac{l_c^2}{t_c} \sim \frac{\alpha_m}{k_m^2} \sim \frac{\lambda}{\rho c_v}\left(\frac{\tilde{\alpha}_m}{\tilde{k}_m^2}\right) \sim a\left(\frac{\tilde{\alpha}_m}{\tilde{k}_m^2}\right), \tag{4.6}$$

where a is the thermal diffusivity which equals $\lambda/\rho c_v$. Here, l_c is the pattern length rather than the thermal diffusion length l_0 which is connected with time t by $(l_0^2/t) \sim a$.

Two interesting special cases are adiabatic deformation and no work hardening.

(i) *Adiabatic conditions*, $\lambda = 0$

In this case the spectral equation (3.10) has the form

$$\rho^2 c_v \alpha^2 + \rho[KP_0\dot{\gamma}_0 + c_v R_0 k^2]\alpha - [K\tau_0 P_0 - \rho c_v Q_0]k^2 = 0. \tag{4.7}$$

If $K\tau_0 P_0 > \rho c_v Q_0$, namely $\mathrm{B} > 1$, it is certain that α has a positive root and instability must occur. But the equation $(\mathrm{d}^2\alpha/\mathrm{d}k^2) = 0$ leads to

$$\alpha_m = \frac{Q_0}{R_0}(\mathrm{B}-1), \tag{4.8}$$

$$k_m = \infty. \tag{4.9}$$

This means that the characteristic length approaches zero in the adiabatic case.

It is important to appreciate that the same formal criterion (4.2) can be used whether the instability is adiabatic or not.

(ii) *No work hardening*, $Q_0 = 0$

The spectral equation (3.10) becomes

$$\rho^2 c_v \alpha^2 + \rho[KP_0\dot{\gamma}_0 + (\lambda + c_v R_0)k^2]\alpha + (\lambda R_0 k^2 - K\tau_0 P_0)k^2 = 0. \tag{4.10}$$

If

$$K\tau_0 P_0 > \lambda R_0 k^2, \tag{4.11}$$

α must have a positive real root and instability will occur. In this criterion the product of the thermal conductivity, the strain-rate hardening and the square of the wave number plays the role which work hardening played previously in (4.2). The equation $(\mathrm{d}^2\alpha/\mathrm{d}k^2) = 0$ leads to

$$k_m^2 = \frac{K\tau_0 P_0 - \rho(\lambda + c_v R_0)\alpha_m}{2\lambda R_0} \tag{4.12}$$

and

$$0 \leqslant \alpha_m \leqslant \frac{K\tau_0 P_0}{\rho(\lambda + c_v R_0)}. \tag{4.13}$$

Combining equations (4.10) and (4.12), we have the following equation for α_m.

$$4R_0\lambda\left(c_v\alpha_m + \frac{KP_0\dot{\gamma}_0}{\rho}\right)\alpha_m = \left[\frac{K\tau_0 P_0}{\rho} - (\lambda + c_v R_0)\alpha_m\right]^2. \tag{4.14}$$

There must be a solution for α_m between

$$0 \quad \text{and} \quad \alpha_m^* = \frac{K\tau_0 P_0}{\rho(\lambda + c_v R_0)},$$

satisfying the instability criterion (4.11).

In this case, the characteristic time is

$$t_c \sim \frac{1}{\alpha_m^*} \sim \frac{\rho(\lambda + c_v R_0)}{K\tau_0 P_0} \sim \frac{\rho c_v R_0^*}{\dot{\gamma}_0 K\tau_0 P_0} \qquad (A \gg 1). \tag{4.15}$$

This is the same result as in the case of $Q_0 > 0$.

Now it is proper to outline the instability mode caused only by thermal softening P_0.

Whether $Q_0 > 0$ or $Q_0 = 0$, there exists a type of shear instability which is dominated by thermal softening. The criterion for the instability mode is formulated in the inequality (4.1).

Only in the case of $Q_0 = 0$ is there another criterion in which strain-rate hardening emerges explicitly to resist the occurrence of shear instability, instead of work-hardening.

This instability mode is characterized by a characteristic wavelength and time

$$t_c \sim \frac{R_0^*}{\dot{\gamma}_0} \frac{\rho c_v}{K\tau_0 P_0 - \rho c_v Q_0}.$$

It is clear that strain-rate hardening R_0^* and the current strain rate $\dot{\gamma}_0$ explicitly influence the time t_c. Owing to the wide range of $\dot{\gamma}_0$, it can be imagined that at high strain rates the delay time can be greatly shortened. This may be one of the causes of the common occurrence of shear instability in dynamic loading regimes.

The characteristic time and length are connected by thermal diffusion and this type of instability is theoretically not necessarily adiabatic.

We turn now to the second mode of shear instability, supposing that there is no thermal softening, i.e. $P_0 = 0$. In this case we can formulate the spectral equation:

$$\rho^2 c_v \alpha^3 + \rho(\lambda + c_v R_0)k^2\alpha^2 + (\lambda R_0 k^2 + \rho c_v Q_0)k^2\alpha + \lambda Q_0 k^4 = 0. \tag{4.16}$$

The parameters λ and R_0 must be positive; however this is not so for Q_0. Therefore, $Q_0 < 0$ may become another possible cause of instability even though it is not certain that $Q_0 < 0$ for isothermal deformation. We rewrite the spectral equation (4.16) in the form

$$\rho^2 c_v \alpha^3 + \rho(\lambda + C_v R_0)k^2\alpha^2 + \lambda R_0 k^4 \alpha = \rho c_v |Q_0| k^2 \alpha + \lambda |Q_0| k^4. \tag{4.17}$$

It is easy to see that there must be a solution $\alpha > 0$; therefore flow must be unstable. It is very simple to show that no maximum in α exists and α is a monotonically increasing function of k, with

$$\lim_{k \to 0} \alpha = 0, \tag{4.18}$$

$$\lim_{k \to \infty} \alpha = \frac{|Q_0|}{R_0}, \tag{4.19}$$

$$\lim_{k \to \infty} t = t_{\min} = \frac{R_0^*}{\dot{\gamma}_0 |Q_0|}. \tag{4.20}$$

This implies that the shorter the wavelength, the earlier the occurrence of instability and the shortest characteristic time is dominated by the current strain rate $\dot{\gamma}_0$. This is nearly the same as the time in the first mode.

Nevertheless, this is a totally different instability mode. There is no further criterion except $Q_0 < 0$, with which, as we have seen, are associated no characteristic length and time. There only exists a minimum time t_{min}.

Obviously, there might exist mixed instability modes ($P_0 > 0$, $Q_0 < 0$) also. No further criteria are needed except for $Q_0 < 0$. Using the same procedure, it can be verified that $\tilde{\alpha}_m$ exists when C $\ll$ 1, A $\gg$ 1.

5. Practical Criterion

Now we can concentrate on the instability mode dominated by thermal softening only, but turn to practical considerations.

The first step is to estimate the dimensionless quantities A and C. For most metals, the relevant parameters have the following orders.

$$\left.\begin{array}{ll} \rho \sim 10^1 \text{ g cm}^{-3}, & c_v \sim 10^7 \text{ erg g}^{-1}\,(^\circ\text{C})^{-1}, \\ \lambda \sim 10^0 \text{ cal cm}^{-1}\text{ s}^{-1}\,(^\circ\text{C})^{-1}, & P_0 \sim 10^7 \text{dyn cm}^{-2}\,(^\circ\text{C})^{-1}, \\ Q_0 \sim 10^9 \text{ dyn cm}^{-2}, & R_0^* \sim 10^7 \text{ dyn s cm}^{-2}. \end{array}\right\} \tag{5.1}$$

These lead to

$$\left.\begin{array}{l} \mathrm{A} \sim \dfrac{c_v R_0}{\lambda} \sim \dfrac{10^7}{\dot{\gamma}_0} \gg 1, \\ \mathrm{C} \sim \dfrac{K\lambda P_0 \dot{\gamma}_0}{\rho c_v^2 Q_0} \sim 10^{-10}\,\dot{\gamma}_0 \ll 1, \end{array}\right\} \tag{5.2}$$

where the unit of strain rate is s^{-1}.

This is the base on which we deduced criteria (4.2), (4.3), etc. A $\gg$ 1 and C $\ll$ 1 mean that when $Q_0 > 0$ the effect of heat conduction on the occurrence of shear instability is small and it is possible to neglect it formally.

It is especially useful that criterion (4.2) implies a strain criterion. Recalling $[Q_0]$ = stress/strain, we can easily deduce that the inequality (4.2) is equivalent to a strain criterion. It is desirable to establish a criterion connecting state parameters and material constants on each side of the inequality. Strain, obviously, is a state parameter. So there must exist a dimensionless combination of material constants implying a sort of strain to identify a special resistance of the material to shear instability.

Supposing P_0 = const., we can formally define such a critical strain, regardless of constitutive equations,

$$\gamma_* = \frac{\tau_0}{Q_0} > \gamma_{i1} = \frac{\rho c_v}{KP_0}. \tag{5.3}$$

Certainly, $\gamma_* = \tau_0/Q_0$ is by no means a true strain, but its meaning is obvious physically and geometrically from Fig. 2.

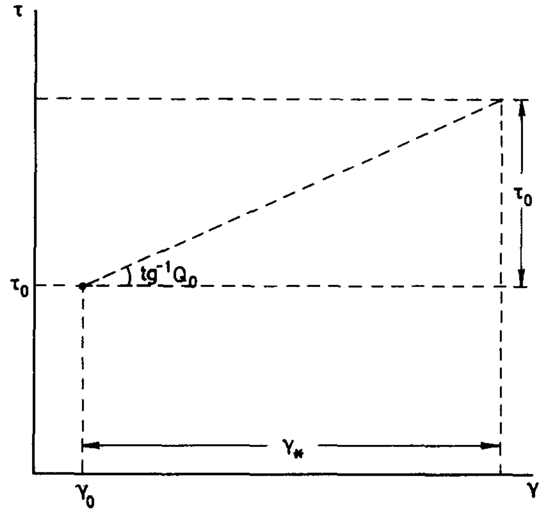

FIG. 2. Graphical interpretation of the parameter γ_*, defined in equation (5.3).

If the stress–strain curve is convex and the work hardening $Q_0 > 0$, γ_* must be an increasing function of γ_0, because τ_0 increases and Q_0 decreases with increasing γ_0. Therefore, at a certain strain, (5.3) will be satisfied and instability occurs. Of course, here γ_{i1} is rather large, so, if the material constants ρ, c_v, K and P_0 are on hand and the relationship of τ and γ is known numerically or graphically, this is a convenient way to judge the occurrence of shear instability.

Because $\gamma_* = (\tau_0/Q_0)$ is not a true strain or a genuine state parameter—on the contrary, γ_* is implicitly controlled by the constitutive relations of materials—it is desirable to derive a genuine critical strain. If the constitutive relation of the material concerned is formulated explicitly, the critical strain is easy to obtain. Suppose, for instance,

$$\tau = G\gamma^n \tag{5.4}$$

where G and n are material constants. Then, the critical strain is

$$\gamma_0 > \gamma_{i2} = \frac{n\rho c_v}{KP_0}. \tag{5.5}$$

If

$$\tau = \tau_Y + G\gamma, \tag{5.6}$$

then

$$\gamma_0 > \gamma_{i3} = \frac{\rho c_v}{KP_0} - \frac{\tau_Y}{G}. \tag{5.7}$$

For most metals, the critical strain for the occurrence of shear instability may be approx. unity or less (CULVER, 1973).

6. Conclusions

It has been shown that there may exist two possible instabilities in simple shear deformation. One is dominated by thermal softening, the other by "work-softening". For the former there is a characteristic time and length.

Generally, the criterion for the first mode of shear instability combines thermal softening, current stress, density, specific heat and work-hardening, and is not related to heat conduction explicitly. Thermal diffusivity connects the characteristic time and length directly in this instability phenomenon. Strain-rate hardening and current strain rate strongly affect the time during which the instability develops fully.

The above criterion implies a practical critical strain. When the constitutive relation has an explicit expression, the critical strain can be described simply.

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